Hopf algebras in Dyson-Schwinger equations

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History

1941 H. Hopf, “Über die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen”.
Cohomology of groups.

1987 S.L. Woronowicz, “Compact Matrix Pseudogroups”.
Quantum groups, integrable systems, non-commutative field theories.

1998 D. Kreimer, A. Connes, “On the Hopf algebra structure of perturbative quantum field theory”.
Perturbative quantum field theory.

Hopf algebras

- An algebra has a multiplication $\cdot$ and a unit e.
  - Multiplication: $A \otimes A \to A$
  - Unit: $K \to A$

- A coalgebra has a comultiplication Δ and a counit $\bar{e}$.
  - Comultiplication: $A \to A \otimes A$
  - Counit: $A \to K$

- A bialgebra is an algebra and a coalgebra at the same time, such that the two structures are compatible with each other.

- A Hopf algebra is a bialgebra. In addition, there is an antipode S.

Axiomatic description of a Hopf algebra

A coalgebra is an algebra with all arrows reversed, for example:

Associativity

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mathrm{id}\otimes\cdot} & A\otimes A \\ \downarrow{\scriptstyle \cdot\otimes\mathrm{id}} & & \downarrow{\scriptstyle \cdot} \\ A\otimes A & \xrightarrow{\cdot} & A \end{array}$$

Coassociativity

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A\otimes A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\ A\otimes A & \xrightarrow{\mathrm{id}\otimes\Delta} & A\otimes A\otimes A \end{array}$$

For a bialgebra the multiplication is compatible with the comultiplication:

$$\begin{array}{ccccc} A\otimes A & \xrightarrow{\cdot} & A & \xrightarrow{\Delta} & A\otimes A \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \uparrow{\scriptstyle \cdot\otimes\cdot} \\ A\otimes A\otimes A\otimes A & & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & & A\otimes A\otimes A\otimes A \end{array}$$

A Hopf algebra is a bialgebra and has in addition an antipode:

$$\begin{array}{ccccc} A & \xrightarrow{\bar{e}} & K & \xrightarrow{e} & A \\ \downarrow{\scriptstyle \Delta} & & & & \uparrow{\scriptstyle \cdot} \\ A\otimes A & & \xrightarrow[S\otimes\mathrm{id}]{\mathrm{id}\otimes S} & & A\otimes A \end{array}$$

Sweedler's notation

The general form of the coproduct is

$$\Delta(t) \quad = \quad \sum_i t_i^{(1)} \otimes t_i^{(2)}.$$

Sweedler's notation consists in dropping the dummy index i and the summation symbol:

$$\Delta(t) \quad = \quad t^{(1)} \otimes t^{(2)}$$

Using Sweedler's notation, the compatibility between the algebra and coalgebra structure is expressed as

$$\left(t_1^{(1)} \cdot t_2^{(1)}\right) \otimes \left(t_1^{(2)} \cdot t_2^{(2)}\right) \quad = \quad \Delta\left(t_1 \cdot t_2\right),$$

and the equation for the antipode reads

$$t^{(1)} \cdot S\left(t^{(2)}\right) = S\left(t^{(1)}\right) \cdot t^{(2)} = 0 \qquad \text{for } t \neq e.$$

Example 0: The group algebra

Let G be a group and denote by KG the vectorspace with basis G. KG is an algebra with the multiplication given by the group multiplication.

The counit ε is given by: $\varepsilon(g) = e.$

The coproduct Δ is given by: $\Delta(g) = g \otimes g.$

The antipode S is given by: $S(g) = g^{-1}.$

KG is a cocommutative Hopf algebra.

KG is commutative if G commutative.

Example I: Lie algebras

A Lie algebra $\mathfrak{g}$ is not necessarily associative nor does it have a unit. To overcome this obstacle one considers the universal enveloping algebra $U(\mathfrak{g})$, obtained from the tensor algebra $T(\mathfrak{g})$ by factoring out the ideal

$$X \otimes Y - Y \otimes X - [X,Y]$$

The coproduct Δ is given by:

$$\begin{aligned} \Delta(e) &= e \otimes e, \\ \Delta(X) &= X \otimes e + e \otimes X. \end{aligned}$$

The antipode S is given by:

$$\begin{aligned} S(e) &= e, \\ S(X) &= -X. \end{aligned}$$

Example II: Quantum SU(2)

The Lie algebra $su(2)$ is generated by three generators H, $X_\pm$ with

$$\begin{aligned} [H, X_\pm] &= \pm 2X_\pm, \\ [X_+, X_-] &= H. \end{aligned}$$

To obtain the deformed algebra $U_q(su(2))$, the last relation is replaced with

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

The coproduct Δ is given by:

$$\begin{aligned} \Delta(H) &= H \otimes e + e \otimes H, \\ \Delta(X_\pm) &= X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm. \end{aligned}$$

The antipode S is given by:

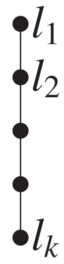
$$\begin{aligned} S(H) &= -H, \\ S(X_\pm) &= -q^{\pm 1} X_\pm. \end{aligned}$$

Example III: Shuffle algebras

Consider a set of letters A. A word is an ordered sequence of letters:

$$w = l_1 l_2 \ldots l_k$$

$w =$



The word of length zero is denoted by e. A shuffle algebra $\mathcal{A}$ on the vector space of words is defined by

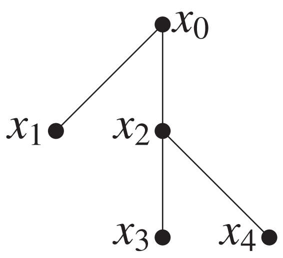
$$(l_1 l_2 \ldots l_k) \cdot (l_{k+1} \ldots l_r) = \sum_{\text{shuffles } \sigma} l_{\sigma(1)} l_{\sigma(2)} \ldots l_{\sigma(r)}$$

where the sum runs over all permutations σ, which preserve the relative order of $1, 2, \ldots, k$ and of $k+1, \ldots, r$.

The coproduct Δ is given by: $$\Delta(l_1 l_2 \ldots l_k) = \sum_{j=0}^{k} (l_{j+1} \ldots l_k) \otimes (l_1 \ldots l_j) .$$

The antipode S is given by: $$S(l_1 l_2 \ldots l_k) = (-1)^k \, l_k l_{k-1} \ldots l_2 l_1 .$$

Example IV: Rooted trees



An admissible cut of a rooted tree is any assignment of cuts such that any path from any vertex of the tree to the root has at most one elementary cut. An admissible cut maps a tree t to a monomial in trees $t_1 \circ \ldots \circ t_{n+1}$. Precisely one of these subtrees t_j will contain the root of t. We denote this distinguished tree by $R^C(t)$, and the monomial delivered by the n other factors by $P^C(t)$.

The counit $\bar{e}$ is given by:

$$\begin{aligned} \bar{e}(e) &= 1, \\ \bar{e}(t) &= 0, \quad t \neq e. \end{aligned}$$

The coproduct Δ is given by:

$$\begin{aligned} \Delta(e) &= e \otimes e, \\ \Delta(t) &= t \otimes e + e \otimes t + \sum_{\text{adm. cuts } C \text{ of } t} P^C(t) \otimes R^C(t). \end{aligned}$$

The antipode S is given by:

$$\begin{aligned} S(e) &= e, \\ S(t) &= -t - \sum_{\text{adm. cuts } C \text{ of } t} S\left(P^C(t)\right) \circ R^C(t). \end{aligned}$$

Commutative versus non-commutative

- Commutative and cocommutative:
  - Group algebra of a commutative group
- Non-commutative and cocommutative:
  - Group algebra of a non-commutative group
  - Universal enveloping algebra of a Lie algebra
- Commutative and non-cocommutative:
  - Shuffle algebra
  - Algebra of rooted trees
- Non-commutative and non-cocommutative:
  - q-deformed algebras

Hopf algebras and renormalisation

Hopf algebras and renormalisation of quantum field theories

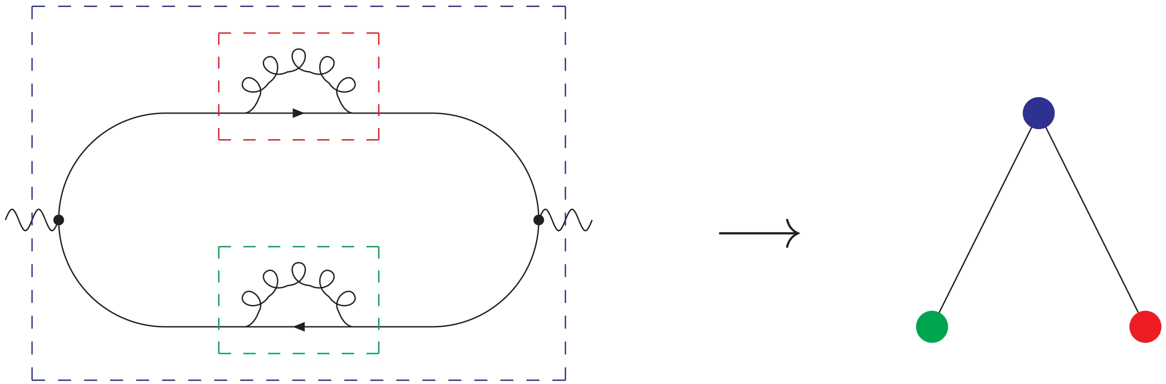
Short-distance singularities of the perturbative expansion of quantum field theories require renormalisation (Bogoliubov, Parasuik, Hepp, Zimmermann).

The combinatorics involved in the renormalisation is governed by a Hopf algebra (Kreimer, Connes).

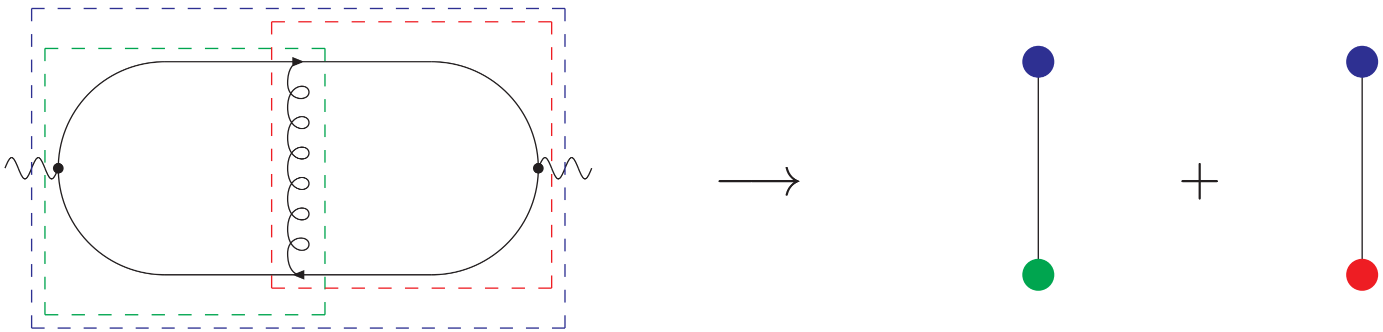
The model for this Hopf algebra is the Hopf algebra of rooted trees.

Subdivergences and rooted trees

Nested singularities are encoded in rooted trees:



Overlapping singularities yield a sum of rooted trees:



Renormalisation

Recall the recursive definition of the antipode:

$$S(t) \quad = \quad -t - \sum_{\text{adm. cuts } C \text{ of } t} S\left(P^C(t)\right) \times R^C(t).$$

The antipode satiesfies

$$m\left[(\mathcal{S} \otimes \text{id})\, \Delta(t)\right] \quad = \quad 0.$$

Let $\mathcal{R}$ be an operation which approximates a tree by another tree with the same singularity structure. Twist the antipode with $\mathcal{R}$ and define a new map

$$\mathcal{S}_{\mathcal{R}}(t) \quad = \quad -\mathcal{R}\left(t + \sum_{\text{adm. cuts } C \text{ of } t} \mathcal{S}_{\mathcal{R}}\left(P^C(t)\right) \times R^C(t) \right).$$

Then

$$m\left[(\mathcal{S}_{\mathcal{R}} \otimes \text{id})\, \Delta(t)\right] \quad = \quad \text{finite.}$$

Renormalisation

$\mathcal{R}$ is not unique and different choices for $\mathcal{R}$ correspond to different renormalisation prescription.
$\mathcal{R}$ has to fulfill the Rota-Baxter relation:

$$\mathcal{R}(t_1 t_2) + \mathcal{R}(t_1)\,\mathcal{R}(t_2) \quad = \quad \mathcal{R}(t_1 \mathcal{R}(t_2)) + \mathcal{R}(\mathcal{R}(t_1)\,t_2)\,.$$

From this multiplicativity constraint it follows that

$$\mathcal{S}_{\mathcal{R}}(t_1 t_2) \quad = \quad \mathcal{S}_{\mathcal{R}}(t_1)\,\mathcal{S}_{\mathcal{R}}(t_2)\,.$$

Minimal subtraction ($\overline{MS}$) fulfills the Rota-Baxter relation

$$\mathcal{R}\left(\sum_{k=-L}^{\infty} c_k \varepsilon^k\right) \quad = \quad \sum_{k=-L}^{-1} c_k \varepsilon^k.$$

Applications in physics

Hopf algebras and loop integrals

Feynman integrals

A Feynman graph with m external lines, n internal lines and l loops corresponds (up to prefactors) in D space-time dimensions to the Feynman integral

$$I_G = \frac{\left(\mu^2\right)^{n-lD/2}}{\Gamma(n-lD/2)} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{(-q_j^2+m_j^2)}$$

The momenta flowing through the internal lines can be expressed through the independent loop momenta k_1, ..., k_l and the external momenta p_1, ..., p_m as

$$q_i = \sum_{j=1}^{l} \lambda_{ij} k_j + \sum_{j=1}^{m} \sigma_{ij} p_j, \qquad \lambda_{ij}, \sigma_{ij} \in \{-1, 0, 1\}.$$

Feynman parametrisation

The Feynman trick:

$$\prod_{j=1}^{n} \frac{1}{P_j} = \Gamma(n) \int_{x_j \geq 0} d^n x \, \delta(1 - \sum_{j=1}^{n} x_j) \frac{1}{\left(\sum_{j=1}^{n} x_j P_j \right)^n}$$

We use this formula with $P_j = -q_j^2 + m_j^2$.
We can write

$$\sum_{j=1}^{n} x_j(-q_j^2 + m_j^2) = -\sum_{r=1}^{l} \sum_{s=1}^{l} k_r M_{rs} k_s + \sum_{r=1}^{l} 2k_r \cdot Q_r + J,$$

where M is a $l \times l$ matrix with scalar entries and Q is a l-vector with momenta vectors as entries.

Feynman integrals

After Feynman parametrisation the integrals over the loop momenta k_1, ..., k_l can be done:

$$I_G = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{n-(l+1)D/2}}{\mathcal{F}^{n-lD/2}}, \qquad \mathcal{U} = \det(M),$$

$$\mathcal{F} = \det(M) \left(J + Q M^{-1} Q \right) / \mu^2.$$

The functions $\mathcal{U}$ and $\mathcal{F}$ are called the first and second graph polynomial.

$\mathcal{U}$ is positive definite inside the integration region and positive semi-definite on the boundary.

$\mathcal{F}$ depends on the masses m_i^2 and the momenta $(p_{i_1} + ... + p_{i_r})^2$. In the euclidean region $\mathcal{F}$ is also positive definite inside the integration region and positive semi-definite on the boundary.

Remarks

$$I_G = \Gamma(n - lD/2) \int_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{n-(l+1)D/2}}{\mathcal{F}^{n-lD/2}},$$

- The integral over the Feynman parameters is a $(n-1)$-dimensional integral, where n is the number of internal edges of the graph.

- The dimension D of space-time enters only in the exponent of the integrand.

- Singularities may arise if the zero sets of $\mathcal{U}$ and $\mathcal{F}$ intersect the region of integration.

- The exponent acts as a regularisation.

- Laurent expansion in $\varepsilon = (4-D)/2$:

$$I_G = \sum_{j=-2l}^{\infty} c_j \varepsilon^j.$$

One-loop amplitudes

All one-loop amplitudes can be expressed as a sum of algebraic functions of the spinor products and masses times two transcendental functions, whose arguments are again algebraic functions of the spinor products and the masses.

The two transcendental functions are the logarithm and the dilogarithm:

$$\begin{aligned} \text{Li}_1(x) &= -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \\ \text{Li}_2(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^2} \end{aligned}$$

Generalisations of the logarithm

Beyond one-loop, at least the following generalisations occur:

Polylogarithms:

$$\text{Li}_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m}$$

Multiple polylogarithms (Goncharov 1998):

$$\text{Li}_{m_1,m_2,\dots,m_k}(x_1,x_2,\dots,x_k) = \sum_{n_1>n_2>\dots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \dots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

This is a nested sum:

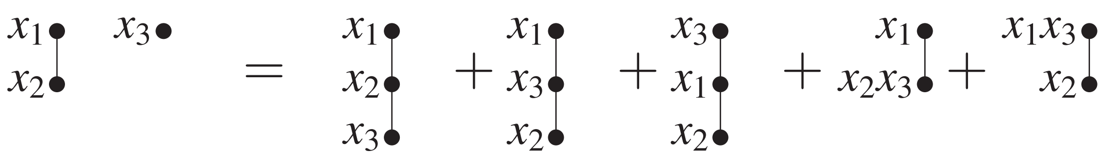
$$\dots \sum_{n_j=1}^{n_{j-1}-1} \frac{x_j^{n_j}}{n_j^{m_j}} \sum_{n_{j+1}=1}^{n_j-1} \dots$$

Multiplication

Multiple polylogarithms obey an algebra:

$$\mathrm{Li}_{m_1,m_2}(x_1,x_2) \cdot \mathrm{Li}_{m_3}(x_3) =$$
$$= \mathrm{Li}_{m_1,m_2,m_3}(x_1,x_2,x_3) + \mathrm{Li}_{m_1,m_3,m_2}(x_1,x_3,x_2) + \mathrm{Li}_{m_3,m_1,m_2}(x_3,x_1,x_2)$$
$$+\mathrm{Li}_{m_1,m_2+m_3}(x_1,x_2x_3) + \mathrm{Li}_{m_1+m_3,m_2}(x_1x_3,x_2)$$

Pictorial representation:



The multiplication law corresponds to a quasi-shuffle algebra (Hoffman '99), also called stuffle algebra (Broadhurst), mixed shuffle algebra (Guo) or mould symmetrel (Ecalle).

Iterated integrals

Define the functions G by

$$G(z_1,...,z_k;y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}.$$

Scaling relation:

$$G(z_1,...,z_k;y) = G(xz_1,...,xz_k;xy)$$

Short hand notation:

$$G_{m_1,...,m_k}(z_1,...,z_k;y) = G(\underbrace{0,...,0}_{m_1-1},z_1,...,z_{k-1},\underbrace{0...,0}_{m_k-1},z_k;y)$$

Conversion to multiple polylogarithms:

$$\text{Li}_{m_1,...,m_k}(x_1,...,x_k) = (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},...,\frac{1}{x_1...x_k};1\right).$$

Shuffle algebra

The functions $G(z_1, ..., z_k; y)$ fulfill a shuffle algebra.

Example:

$$G(z_1, z_2; y) G(z_3; y) = G(z_1, z_2, z_3; y) + G(z_1, z_3, z_2; y) + G(z_3, z_1, z_2; y)$$

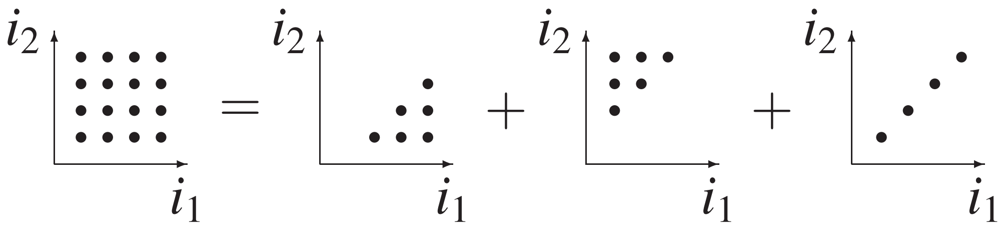
This algebra is different from the quasi-shuffle algebra already encountered and provides the second Hopf algebra for multiple polylogarithms.

A shuffle algebra is also called a mould symmetral (Ecalle).

Shuffle algebra versus quasi-shuffle algebra

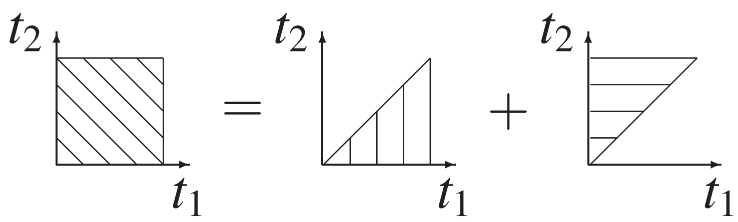
Quasi-shuffle algebra from the sum representation:

$$\text{Li}_{m_1}(x_1)\text{Li}_{m_2}(x_2) = \text{Li}_{m_1,m_2}(x_1,x_2) + \text{Li}_{m_2,m_1}(x_2,x_1) + \text{Li}_{m_1+m_2}(x_1x_2).$$



Shuffle algebra from the integral representation:

$$G(z_1;y)G(z_2;y) = G(z_1,z_2;y) + G(z_2,z_1;y)$$



Summary on multiple polylogarithms

Nice features:

- Many Feynman integrals evaluate to multiple polylogarithms
- Multiple polylogarithms obey two Hopf algebras.

Caveats:

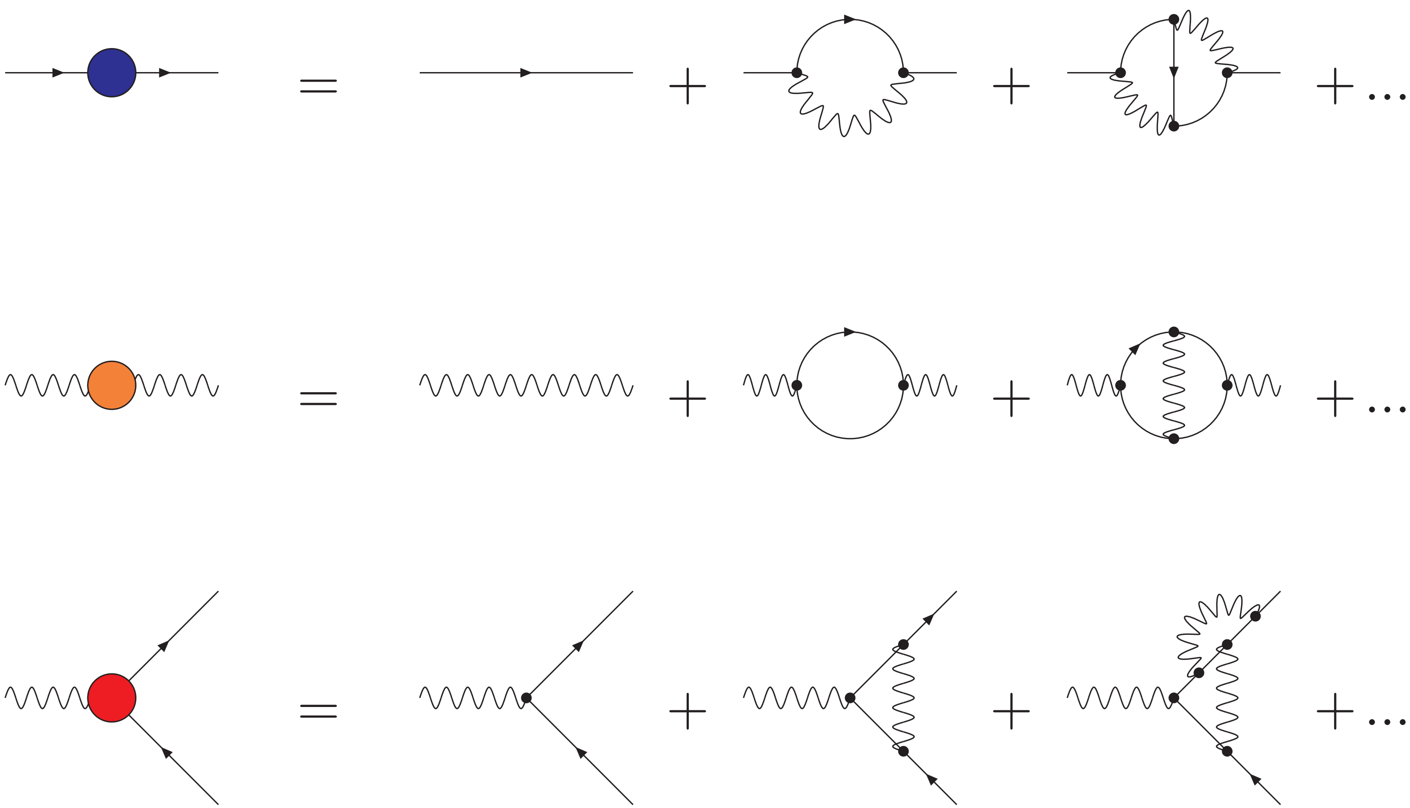
- No Hopf algebra homomorphism from the algebra of Feynman graphs to the algebra of multiple polylogarithms.
- There are graphs which cannot be expressed in terms of multiple polylogarithms.

Part III

Dyson-Schwinger equations

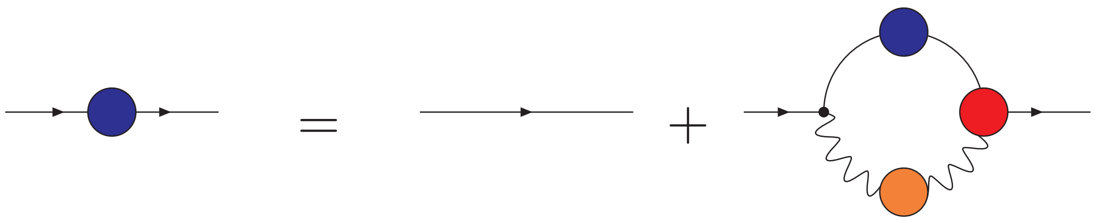
Feynman diagrams

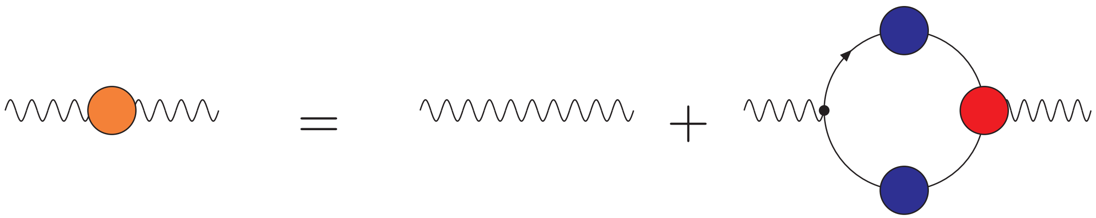
Convention: Let a blob represent all possible Feynman diagrams with a given number of external lines.



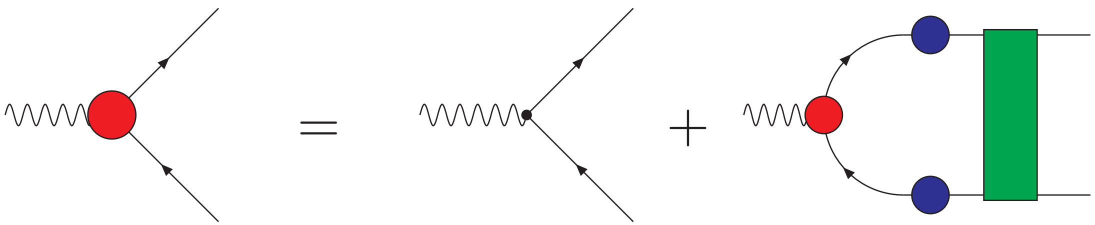
Dyson-Schwinger equations

Propagators:





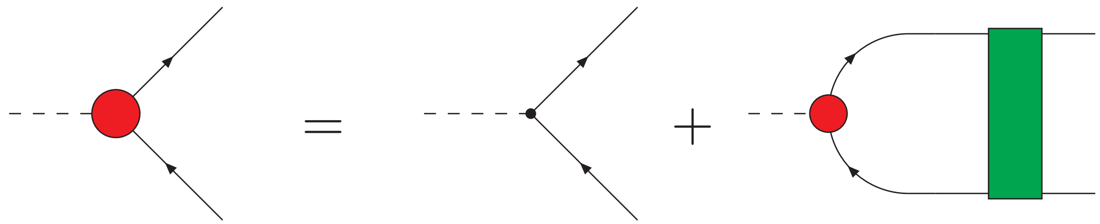
Vertex:



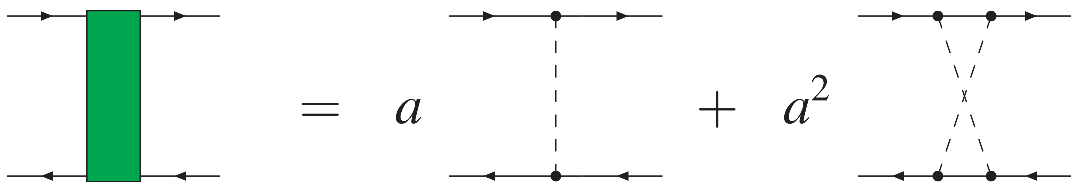
Example: Simplified Dyson-Schwinger equation

- Toy model with a fermion and a scalar.

- Linearise Dyson-Schwinger equation:



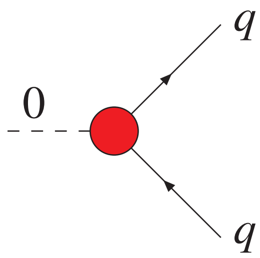
- Truncate kernel at two-loop order:



Example: Details

Simplify kinematics (scalar has zero momentum):

$G_R(a,L) =$



with

$$a = \text{coupling}$$
$$L = \ln\left(\frac{-q^2}{\mu^2}\right)$$

Renormalisation condition:

$$G_R(a,0) = 1.$$

Dimensional analysis:

$$G_R(a,L) = \exp(-\gamma_G(a)L).$$

Anomalous dimension γ_G depends only on coupling a, but not on L.

Example: Calculation

From the Dyson-Schwinger equation we obtain

$$\exp\left(-\gamma_G(a)L\right) \quad = \quad 1 + \left(\exp\left(-\gamma_G(a)L\right) - 1\right)\left[aF_1(\gamma_G) + a^2F_2(\gamma_G)\right],$$

where F_1 and F_2 are the Mellin-transforms of the one-loop and two-loop integral, respectively. Working these out, we find

$$1 = -a\frac{1}{\gamma_G(1-\gamma_G)} - a^2\left\{\frac{1}{\gamma_G^2(1-\gamma_G)^2} - 4\sum_{n=1}^{\infty} n(1-2^{-2n})\zeta_{2n+1}\left[\gamma_G^{2n-2} + (1-\gamma_G)^{2n-2}\right]\right\}.$$

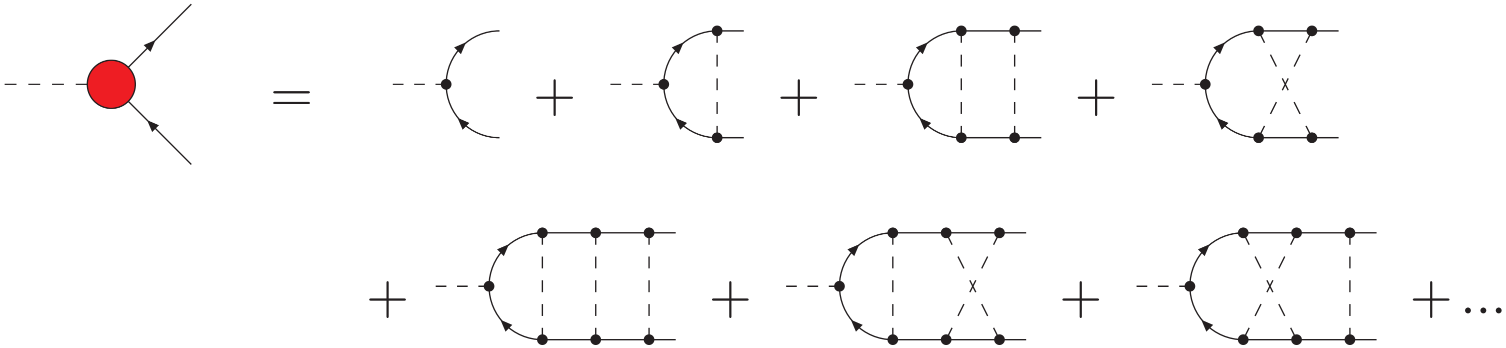
The sum can be done:

$$\begin{aligned}
&-4\sum_{n=1}^{\infty} n(1-2^{-2n})\zeta_{2n+1}\left[\gamma_G^{2n-2} + (1-\gamma_G)^{2n-2}\right] = \\
&= \quad \frac{1}{\gamma_G}\left[\psi'(1+\gamma_G) - \psi'(1-\gamma_G)\right] + \frac{1}{1-\gamma_G}\left[\psi'(2-\gamma_G) - \psi'(\gamma_G)\right] \\
&\qquad -\frac{1}{2\gamma_G}\left[\psi'\left(1+\frac{\gamma_G}{2}\right) - \psi'\left(1-\frac{\gamma_G}{2}\right)\right] - \frac{1}{2(1-\gamma_G)}\left[\psi'\left(\frac{3-\gamma_G}{2}\right) - \psi'\left(\frac{1+\gamma_G}{2}\right)\right].
\end{aligned}$$

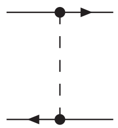
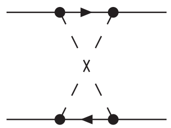
This defines implicitly γ_G as a function of a.

The Hopf side of the example

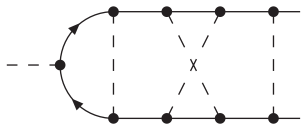
The Dyson-Schwinger equation computes



Introduce two letters u and v:

$u =$  $v =$ 

Can represent any diagram by a word, for example

$uvu =$ 

The Hopf side of the example

Define insertion operators B_+^u, B_+^v by the action on any words w

$$B_+^u w = uw, \qquad B_+^v w = vw.$$

Re-write Dyson-Schwinger equation as

$$X(a) \quad = \quad 1 + aB_+^u X(a) + a^2 B_+^v X(a)$$

Then

$$X(a) \quad = \quad \exp_{\text{Ш}}\left(au + a^2 v\right),$$

where Ш denotes the shuffle product:

$$u^{\text{Ш}n} = \underbrace{u \,\text{Ш}\, u \,\text{Ш}\, \ldots \,\text{Ш}\, u}_{n} = n!\,\underbrace{uu\ldots u}_{n}, \qquad u\,\text{Ш}\,v = uv + vu.$$

Thus

$$X(a) \quad = \quad 1 + au + a^2(uu + v) + a^3(uuu + uv + vu) + \ldots$$

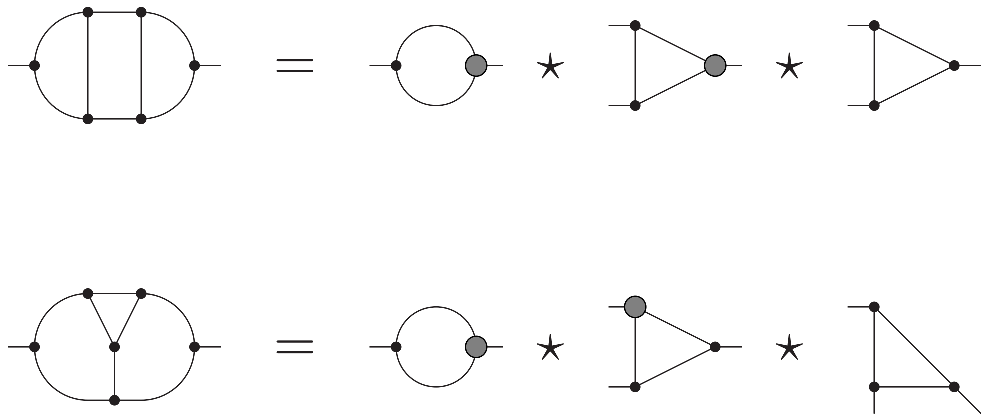
I. Bierenbaum, D. Kreimer, S.W.

Generalisations

Combinatorial Dyson-Schwinger equations:

- Map iterated structure of graphs to iterated structure of Feynman integrals:
  - Mellin-Barnes
  - Nested sums
  - Linear reducibility

- Mellin convolution product:



Conclusions

- Hopf algebras organise combinatorial issues in physics.
- Important examples
  - Renormalisation
  - Algebra of transcendental functions, like multiple polylogarithms
- The combinatorial part of Dyson-Schwinger equations is described by Hopf algebras.