# Solving a Dyson–Schwinger equation around its first singularity in the Borel plane

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Join work with Marc P. Bellon.

**Abstract**

The Dyson–Schwinger equation of the massless Wess–Zumino model is written as an equation over the anomalous dimension of the theory. The asymptotic of its behavior is derived and the procedure to compute perturbations of this asymptotic behavior is detailled. This procedure makes use of ill-defined objects. To cure this, the Schwinger–Dyson equation is rephrased in the Borel plane. It is shown that the ill-defined procedure in the physical plane has a meaning in the Borel plane. Other results obtained in the Borel plane are stated, one of them with a sketch of proof.

## Introduction

In this talk I will present our procedure to deal with the Dyson–Schwinger equation of the massless Wess–Zumino model. This model is clearly a toy model, but is already interesting since its Dyson–Schwinger equation is non linear. When working with this equation, we have two goals:

- to reach precise non-perturbative informations about the model (i.e. beyond the asymptotic behavior of its anomalous dimension),
- to have a well defined procedure.

The second point is important since we are working with a toy model. We want to be sure that every point of the computation is solid before tackling more physically relevant theories.

I will start by introducing the Dyson–Schwinger equation we will be dealing with. In the second part, our procedure of computing the perturbations of the asymptotic behavior of the anomalous dimension of the model will be presented. After a quick introduction of the Borel transform it will be explained in the fourth part how this procedure is reinterpreted in the Borel plane and how the Borel plane approach allows to tackle questions very technical to address in the physical plane.

## 1 Dyson–Schwinger Equation of the Wess–Zumino model

We work with the massless Wess–Zumino model. Therefore, it can be shown that the 2-points function can be expand as a series of the logarithm of the external impulsion:

$$G(L) = 1 + \sum_{k=1}^{+\infty} \gamma_k \frac{L^k}{k!} \tag{1}$$

with $L = \ln(p^2/\mu^2)$. Moreover, the $\gamma_n$ are themselves function of $a$, the fine structure of the theory: $\gamma_n = \gamma_n(a)$. It has been shown in the thesis [1] and in the article [2] that those coefficients are linked by the renormalisation group equation:

$$\gamma_{k+1} = \gamma(1 + 3a\partial_a)\gamma_k \tag{2}$$

with $\gamma := \gamma_1$ being the anomalous dimension of the theory. Hence it is enough to know $\gamma$ to fully know the dressed 2-points function: we will write the Dyson–Schwinger equation as an equation over $\gamma$ rather than an equation over $G(L)$.

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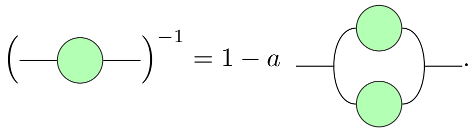
Now, the Dyson–Schwinger equation of the Wess–Zumino model is simply:



(3)

We are working in an exactly supersymmetric theory. Then, using superspace techniques, it can be shown that there is no vertex renormalisation. This is why we will use the singular through this talk when speaking of the Dyson–Schwinger equation: there is no infinite tower of equations in this theory. It makes it very suitable for developping techniques for Dyson–Schwinger equations in general. Now, the loop integral in the right-hand-side of the equation is

$$\int \mathrm{d}^4 q \frac{G(q^2)}{q^2} \frac{G((p-q)^2)}{(p-q)^2}.$$

It can be evaluated by perfoming a Mellin transform. This is nothing but noticing

$$\left(\ln \frac{p^2}{\mu^2}\right)^k = \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^k \left(\frac{p^2}{\mu^2}\right)^x \Bigg|_{x=0}.$$

Then, after permuting sums, derivatives and integral, taking a derivative with respect to $L$ and having evaluated it at $L = 0$ we hand up with the following equation:

$$\gamma = a\Big(1 + \sum_{n=1}^{+\infty} \frac{\gamma_n}{n!} \frac{\mathrm{d}^n}{\mathrm{dx}^n}\Big)\Big(1 + \sum_{m=1}^{+\infty} \frac{\gamma_m}{m!} \frac{\mathrm{d}^m}{\mathrm{dx}^m}\Big) H(x,y)|_{x=y=0} := \mathcal{I}(H). \tag{4}$$

With $H(x,y)$ the function known as the one loop Mellin transform

$$H(x,y) = \frac{\Gamma(1-x-y)\Gamma(1+x)\Gamma(1+y)}{\Gamma(2+x+y)\Gamma(1-x)\Gamma(1-y)} \tag{5}$$

with $\Gamma$ the usual Euler's gamma function. The equation (4) has to be solved together with the renormalisation group equation (2). Hence, it is an equation over $\gamma$ only.

## 2 Solving the Dyson–Schwinger equation

Our goal is to compute the asymptotic behavior of the solution of (4), and then its perturbations. So, we can replace the one loop Mellin transform by a suitable truncation. In order to do so, we will replace $H(x,y)$ by a sum over its poles times its residues. From its definition, it is clear to see that $H(x,y)$ has a pole if and only if an argument of its numerator is a negative integer.

$$H(x,y) \longrightarrow \sum_{k\geq 1} \left[\frac{\mathrm{Res}_{x=-k}(H)}{k+x} + \frac{\mathrm{Res}_{y=-k}(H)}{k+y} + \frac{\mathrm{Res}_{x+y=k}(H)}{k-x-y}\right]. \tag{6}$$

This somehow break down the $x \leftrightarrow y$ symmetry of $H$. We restore it (since it simplifies the computations) by taking a analytic extension of the residues.

$$\begin{aligned} \mathrm{Res}_{x=-k}(H) &\longrightarrow P_k(xy) \\ \mathrm{Res}_{x+y=k}(H) &\longrightarrow Q_k(xy) \end{aligned}$$

And similarly for $y$. Now, $P_k(X)$ and $Q_k(X)$ are polynomials that coincide to the residues of $H$ when restricted to $x$ (or $y$)$= -k$ for $P_k$ and to $x + y = k$ for $Q_k$. The general expression for those polynomials is

$$P_k(xy) = \frac{xy}{k(k-1)} \prod_{i=1}^{k} \left(1 + \frac{xy}{ki}\right) \prod_{i=1}^{k-2} \left(1 + \frac{xy}{ki}\right) \tag{7}$$

$$Q_k(xy) = \frac{xy}{k(k+1)} \prod_{i=1}^{k-1} \left(1 - \frac{xy}{i(k-i)}\right) = Q_k(xy) \tag{8}$$

as shown in [3]. Now, it is easy to see to which order of $\gamma$ any given term of the expansion of $H$ will contribute. Hence we will perform such a truncation of 6. To see how this is working in practice, let me look at an example.

**Example:** asymptotic behavior.
Only the two first poles are needed to compute the asymptotic behavior of $\gamma$. Therefore we will replace $H$ by

$$h(x,y) = (1+xy)\left(\frac{1}{1+x}+\frac{1}{1+y}-1\right)+\frac{1}{2}\frac{xy}{1-x-y}+\frac{1}{2}xy. \tag{9}$$

It is useful to look at the quantities defined as the contribution of the singular parts of $H$ (without the residues) to the $\gamma$ function through the equation (4):

$$\begin{aligned} F(a) &= \mathcal{I}\left(\frac{1}{1+x}\right) \\ L(a) &= \mathcal{I}\left(\frac{1}{1-x-y}\right). \end{aligned}$$

Then, the renormalisation group equation (2) could be used to fine equations fulfill by $F$ and $L$. Those intermetdiate quantities can also be plug into the Dyson–Schwinger equation (4). Those computations being done, we end up with the following system of equations to be solved

$$F = 1-\gamma(3a\partial_a+1)F, \tag{10}$$
$$L = \gamma^2+\gamma(3a\partial_a+2)L, \tag{11}$$
$$\gamma = 2aF-a-2a\gamma(F-1)+\frac{1}{2}a(L-\gamma^2). \tag{12}$$

Then, expanding $\gamma$, $F$ and $L$ in power of $a$: $F=\sum f_n a^n$, $L=\sum l_n a^n$, $\gamma=\sum c_n a^n$; assuming that the $(f_n)$, the $(l_n)$ and the $(c_n)$ have a fast growth, and keeping only the most important terms we find, after some manipulations

$$f_{n+1} \simeq -(3n+5)f_n, \tag{13}$$
$$l_{n+1} \simeq 3nl_n, \tag{14}$$
$$c_{n+1} \simeq -(3n+2)c_n. \tag{15}$$

This was a result of [2]. The next natural step would be to compute the $1/n$ corrections of the recursion for the $c_n$. However, it turns out to be quite tricky very soon, mainly due to the fact that every poles will contribute to the $1/n$ order. Moreover, such procedure is not very easy to implement on a formal computational software.

The idea of the article [3] was to use the fact that we know the contributions of $F$ and $L$ to compute the perturbations of the asymptotic behavior of $\gamma$. Hence, let us define two sequences:

$$\begin{aligned} A_{n+1} &= -(3n+5)A_n \\ B_{n+1} &= 3nB_n \end{aligned}$$

and the two formal series

$$\begin{aligned} A &= \sum A_n a^n \\ B &= \sum B_n a^n. \end{aligned}$$

Because of the definition of the sequences $(A_n)$ and $(B_n)$, the formal series have to obey two differential equations

$$3a^2\partial_a A = -A-5aA \tag{16}$$
$$3a^2\partial_a B = B \tag{17}$$

up to a term of degree $n_0$ (the initial condition, i.e. the part of the formal series that is not defined by the inductive definitions). Hence we will have to take $n_0$ bigger that the computed order. Now we can expand $\gamma$ around the formal series $A$ and $B$ by taking the ansatz

$$\gamma(a) = a[c(a)+d(a)A+e(a)B]. \tag{18}$$

We will also make a similar ansatz to any intermediate quantities needed in the computation. Then the general procedure is:

- Write the Dyson–Schwinger equation with the truncated Mellin transform.
- Use the ansatz with $A$ and $B$ and the relations (16) to get rid of the derivatives of $A$ and $B$.
- Ignore the mixed terms ($AB$, $A^2$...).
- Ask for each of the remaining terms to vanish.
- Solve the equations perturbatively, without assuming any fast growth: it was already taken care of in the formal series $A$ and $B$.

This is quite a powerful and algorithmic procedure. It allowed us in [3] to compute the fourth order of the pertubations of the asymptotic behavior of $\gamma$. Moreover, the results showed remarkable agreement with the numerical data of [2]. However, we are not yet fully happy with it for some reasons:

- It relies on analytic continuation of the residues Mellin transform. We used in the computation
$$H(x,y) = \sum_{k\geq 1}\left[\left(\frac{1}{k+x}+\frac{1}{k+y}-\frac{1}{k}\right)P_k(xy)+\frac{Q_k(xy)}{k-x-y}\right],$$
that we could not proved to be exact: an analytic term may be missed by such a expansion over poles. Typically, we saw that the $1/k$ had to be added to have the right value of the derivative at the origin, but we have no meaningful understanding of why it has to be put there. We checked that there is missing analytical terms at the computed orders, and that an eventual analytic term could not be diagonal, however a full analysis is still missing and quite cumbersome.
- The use of formal series make the comparison to numerical results quite unnatural.
- The dropping of the mixed term is questionnable. It is justified by saying that $A$ and $B$ encode very different type of contributions to the asymptotic, and so do not talk to each other. But this is not quite a rigorous argument.
- An analysis at every order is very technical. Even to show that there is no $\zeta(2n)$ in the solution is an highly non-trivial task. We expect that there is no $\zeta(2n)$ in the solution since one can write $H$ as the exponential of a polynomial having odd zetas as coefficients.

Mapping our problem to the Borel plane seems to be relevant to cure (at least) some of this issues.

# 3 Borel transform

We will just write here the definition of the Borel transform that we used and some of its very basic properties. We see the Borel transform as a ring morphism between two rings of formal series:

$$\begin{aligned}\mathcal{B}: a\mathbb{C}[[a]] &\longrightarrow \mathbb{C}[[\xi]] \\ \tilde{f}(a) = a\sum_{n=0}^{+\infty} c_n a^n &\longrightarrow \hat{f}(\xi) = \sum_{n=0}^{+\infty}\frac{c_n}{n!}\xi^n.\end{aligned} \tag{19}$$

Even if $\tilde{f}$ is a purely formal series (i.e. has a null radius of convergence), $\hat{f}$ might be convergent. AnS interesting thing about the Borel transform is that it has an inverse called the Laplace transform. The later matches the usual sum whenever $\tilde{f}$ is convergent, and gives sometimes an analytical meaning to divergent series, in our case to $A$ and $B$.

A caveat has to be done here: in general, this resummation has to be done in sectors of the complex plane, bounded by the singularities of the Borel transform. One speaks of sectorial resummation. When one goes to one sector to an other (i.e. crosses a line between the origin and a singularity of the Borel transform), the result of the summation changes. This is known as the Stokes phenomenon, and its study is very active in the field of dynamical systems.

An important property of the Borel transform is that the Borel transform of a point-like product is the convolution product of the Borel transforms:

$$\begin{aligned}\mathcal{B}(\tilde{f}\tilde{g})(\xi) &= \hat{f}\star\hat{g}(\xi) \\ &= \int_0^{\xi}\hat{f}(\xi-\eta)\hat{g}(\eta)\mathrm{d}\eta.\end{aligned} \tag{20}$$

The last line above being defined if and only if $\hat{f}$ and $\hat{g}$ are two functions. In the following, $\xi$ will be the Borel transform of the fine structure constant $a$. When working with $a$, we will say that we are in the physical plane, to the contrary of the Borel plane (where we are when working with $\xi$).

## 4 Dyson–Schwinger equation in the Borel plane

First, we want to check that the procedure of dealing with the Dyson–Schwinger equation in the physical plane detailled in section 2 is coherent with the Borel plane approach. Hence, let us inductively define a sequence

$$A_{n+1} = (\alpha n - \beta)A_n \tag{21}$$

and the corresponding formal series

$$A = \sum A_n a^n. \tag{22}$$

Now, we can compute the Borel transform of this formal series. It is easy to check that

$$\begin{aligned} \widehat{aA} &\underset{\xi\to 1/\alpha}{\sim} (1-\alpha\xi)^{\beta/\alpha} \quad \text{for } \frac{\beta}{\alpha} \notin \mathbb{N} \\ &\underset{\xi\to 1/\alpha}{\sim} (1-\alpha\xi)^{\beta/\alpha}\ln(1-\alpha\xi) \quad \text{for } \frac{\beta}{\alpha} \in \mathbb{N}. \end{aligned}$$

We have written here the Borel transform of $aA$ rather than the one of $A$ since it is simpler and also because there is no $A$ alone in $\gamma$. Hence, one sees that the $A$ and $B$ formal series translate into the Borel plane as singularities of $\hat{\gamma}$ at $\xi = \pm 1/3$. More precisely:

$$\begin{aligned} \hat{\gamma}(\xi) &\underset{\xi\to -1/3}{\sim} \left(1+\frac{\xi}{3}\right)^{-5/3} \quad \text{for } \frac{\beta}{\alpha} \notin \mathbb{N} \\ &\underset{\xi\to 1/3}{\sim} \ln\left(1-\frac{\xi}{3}\right) \quad \text{for } \frac{\beta}{\alpha} \in \mathbb{N}. \end{aligned}$$

Now, we have to check that ignoring the mixed terms in the physical plane makes sens in the Borel plane. So, let us take two functions similar to those used in the computation in the physical plane.

$$\begin{aligned} f(a) &= a^n + a^m A \\ g(a) &= a^p + a^q A. \end{aligned}$$

The analysis in the physical plane relies on the following replacement

$$f(a)g(a) \longrightarrow a^{n+p} + \left[a^{m+p} + a^{q+n}\right] A. \tag{23}$$

And indeed, it turns out that:

$$\begin{aligned} \hat{f} \star \hat{g}(\xi) &\underset{\xi\to 0}{\sim} \mathcal{B}(a^{q+n}A) \\ &\underset{\xi\to 1/\alpha}{\sim} \mathcal{B}(\left[a^{m+p} + a^{q+n}\right] A). \end{aligned} \tag{24}$$

The first point is trivial (since the Borel transform is well-defined for ordinary functions). For the second case, one has to do the distinguish between $\beta/\alpha \notin \mathbb{N}$ and $\beta/\alpha \in \mathbb{N}$. When doing the computation, one quickly realize that (for both cases) the singular contribution will be the expected one. The main worry is the the combinatorial factors coming during the computation. The fact that they are the same in the two members of the equation is important: if they were not, then the computation in the physical plane would have to be corrected.

Hence, the procedure of [3] is strictly equivalent to solving the Dyson–Schwinger equation in the Borel plane in the vicinity of $\xi = \pm 1/3$. Now, one could ask: what makes those singularities special? To answer that question, we will look for the other singularities of $\hat{\gamma}$.

From now on, I will be even more sketchy than before. The detailled computations will be presentated in the forthcoming article [4]. Let us assume that $\hat{\gamma}$ has an algebraic singularity of order $\alpha$ in $\xi = \xi_0$. This means that $\hat{\gamma}$ diverges at $\xi = \xi_0$ and that this divergence is cancelled by any polynomial (with non-integer power) of degree stricly more than $\alpha$. More rigourously, $\alpha$ is the smallest real number such that

$$\forall \varepsilon > 0, \left|(\xi-\xi_0)^{\alpha+\varepsilon}\hat{\gamma}(\xi)\right| \underset{\xi\to\xi_0}{\longrightarrow} 0.$$

We will write $\mathrm{sing}_{\xi_0}(\hat{\gamma}) = \alpha$. The virtue of that definition is that one has not to worry about logarithms any more. The drawback of this is that if we prove a statement like "$\hat{\gamma}$ has a singularity of order $\alpha$ in $\xi = \xi_0$", then we could have in $\xi = \xi_0$ $\hat{\gamma}(\xi) \sim (\xi - \xi_0)^\alpha$ or $\hat{\gamma}(\xi) \sim (\xi - \xi_0)^\alpha \ln|\xi - \xi_0|$, or any other function divering in $\xi = \xi_0$ more slowly than any polynomials. Actually, it is quite easy to see from the above analysis than logarithms will only occurs when $\alpha$ is a strictly negative integer.

The statement that $\hat{\gamma}$ has only algebraic singularities is quite strong, indeed. But it is justified by the physical plane anaysis: the asymptotic behavior of $\gamma$ was encoded into the singularities of its Borel transform at $\xi = \pm 1/3$, which are algebraic singularities. And a singularity more serious than any algebraic one would be expected to contribute to the asymptotic of $\gamma$.

The procedure to find the localization of the singularities of $\hat{\gamma}$ is first to plug the assumption that it has an algebraic singularity in $\xi = \xi_0$ into the Borel transform of the renormalisation group equation. This tells us how this singularity is transferred to $\hat{G}$, the Borel transform of the two point function. Then we plug this result into the Dyson–Schwinger equation written under the form

$$\hat{\gamma}(\xi) = \mathcal{F}[\hat{G}](\xi) \tag{25}$$

with $\mathcal{F}$ a functionnal that it would be too lengthy to detail here. Then one check at which conditions we have the same kind of singurities in the two members of the equation. The answer is:

$$\mathrm{sing}_{\xi_0}\left(\mathcal{F}[\hat{G}]\right) = \alpha \Leftrightarrow \xi_0 = \frac{k}{3}, \quad k \in \mathbb{Z}^*. \tag{26}$$

Actually, with this procedure we just show the implication of the above statement, but the converse implication is trivial when looking at $\mathcal{F}[\hat{G}](\xi)$.

This shows that the title of this talk is justified: $\xi = \pm 1/3$ are the two first singularities (i.e. the nearest to the origin).

# Conclusion

Now, we have explained how the Borel plane approach of the Dyson–Schwinger equation of the massless Wess–Zumino model sheds light on a procedure otherwise not entirely satisfactory. However, we have also find where all the singularities of $\hat{\gamma}$ lies. This suggests that much more is possible within the Borel plane approach. And indeed, it is!

We have found in [4] relations between the leading coefficients of every singularities. Moreover, the divergence degree of $\hat{\gamma}$ was found to be $\alpha_k = \mathrm{sgn}(k)\frac{2}{3}(|k| - 1)$ for a singurality in $\xi = k/3$ ($k \neq -1$). A link between the combinatorics of Multi-Zeta-Values and the asymptotic behavior of $\hat{\gamma}$ (away from the real line and for positive real values of $\xi$) was also found. It allowed to put a bound on $|\hat{\gamma}|$ over some non-compact sectors of the complex plane.

However, some interesting results about the number content of the expansion of $\hat{\gamma}$ around its first singularities could also be found. For example, it is very easy to show in this framework that no $\zeta(2n)$ will occur. A bound on the weight of the $\zeta(2n+1)$ have also been set. Those are quite teasing results, and we are know seeking for similar ones for the other singularities of $\hat{\gamma}$. Nevertheless, a new complication appears for the higher singularities: the convolution product becomes harder to define since it can whirl around a singularity. Devices have been construct by Jean Écalle to deal with such technicalities: the Alien Calculus. This will be the subject of Marc Bellon's talk tomorrow and I will not say more about it know.

As a final word, I would like to emphasize that the Borel plane approach of Dyson–Schwinger equation allows to extract a lot of informations about its solution. However, it does not allow to do everything. In particular, numerical analysis is much more involved than in the physical plane, due to the convolution integrals that are very sensible to numerical instabilities. Even more important maybe: it is very hard to compute the rationnal factors in front of the zetas arising in the expension of the anomalous dimension around its singularities. Such numbers are not very interesting for the mathematician, but are crucial to the physicist if he wants to ever compare its results to an experiment. Thus, in that sens, Borel plane and physical plane approaches are complementary and both deserve to be investigated further.

# References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