

Combinatorial DSE.

Def Hopf algebra of Rooted trees

$$\mathcal{X} = \mathbb{Q}[\text{rooted tree. } \begin{array}{c} \circ \\ | \\ \downarrow \end{array} \text{ oriented } \downarrow]$$

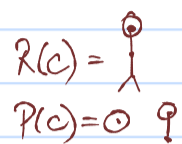
prod. disjoint union
admiss cut. collection of edges, $c \in E(T)$
st any path contains at most 1

• Full cut

• Empty cut

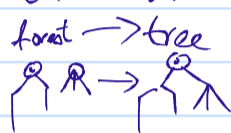


$$\Delta_c(T) = \sum P(c) \otimes R(c)$$



$$\Delta T = \sum_{\text{admiss cuts}} \Delta_c(T)$$

$B_+ \mathcal{X} \rightarrow \mathcal{X}$ Linear operator.



Eg of Combinatorial DSE:

$$\textcircled{1} X = 1 + \alpha B_+(X^2) =$$

$$1 + \alpha \cdot + 2\alpha^2 \cdot + \alpha^3 (2 \cdot + 1 \cdot) + \alpha^4 (2 \cdot + 4 \cdot) + \dots$$

Partial Green's function

$$\Gamma^{\circ} = 1 + \alpha \cdot + 2\alpha^2 \cdot + \alpha^3 (\cdot + \cdot) + \dots$$

In number theory:

$$X = 1 + \alpha B_+(X) = 1 + \alpha \cdot + \alpha^2 \cdot + \alpha^3 \cdot + \dots$$

$$\begin{array}{c} \circ \\ | \\ \downarrow \\ \vdots \end{array} \rightarrow \int_0^z \frac{dt_1}{t_1} \dots \int_0^{t_2} \frac{dt_2}{t_2} \int_0^{t_3} \frac{dt_3}{1-t_3}$$

$$\Rightarrow X = \sum \alpha^n \text{Li}_n(z)$$

More examples, see Manin

Eg computation theory.

Note: $\Delta B_+(t_1, t_2) = \sum_{c \text{ admiss}} P(c) \otimes B_+(R(c)) + B_+(t_1, t_2) \otimes 1$

doesn't take full total cut

Connection to non-commutative geometry.

Hochschild cohomology

$$A = \text{associative alg}$$

$$T(A) = \hat{\bigoplus}_{n \geq 1} A^{\otimes n}$$

Hochschild homology: $\partial: A^{\otimes n} \rightarrow A^{\otimes n-1}$

$$[a_1 \otimes a_2 \otimes \dots \otimes a_n] \rightarrow \sum (-1)^i a_1 \otimes \dots \otimes a_{i-1} \otimes a_{i+1} \otimes \dots \otimes a_n$$

de Rham cohom in non-commutative setting

Hochschild cohom.

$$\Delta^{(i)}: A^{\otimes n} \rightarrow A^{\otimes n+1}$$

$$a_1 \otimes \dots \otimes a_n \rightarrow a_1 \otimes \dots \otimes a_{i-1} \otimes a_{i+1} \otimes \dots \otimes a_n$$

$$T \in C^n$$

$$C^*(B, A) = (\text{Hom}_{i, n} (B, A^{\otimes i}), b)$$

$$bT = (1 \otimes T)\Delta + \sum (-1)^i \Delta^{(i)}(T) + T \otimes 1$$

Thm $B_+ \in HH^2(\mathcal{X}, \mathcal{X})$ (Kreimer, Bergbauer)

Thm (Hochschild-Kostant-Rosenberg)

$HH^1(\mathcal{X}, \mathcal{X}) =$ vector space generated by prim graphs

Def Prim graph: $\Delta t = 1 \otimes t + t \otimes 1$

Eg: $\begin{array}{c} \circ \\ | \\ \downarrow \\ \vdots \end{array} \quad 3 \cdot (1 - \cdot) - \dots$
etc.

However, we know only 1 element!

Q: What others exist?
What is the cohomology?

Where has this been studied?

Connes-Moscovici (98)

$M = 1$ -dim locally param m.fld

$$X: V \rightarrow U$$

$$A = C_c^\infty(F^+(M)) \rtimes D, \text{ff}^+(M)$$

$f \cup_\psi$ $U \subset M$ defines supp f

$$(f \cup_\psi)(g \cup_\psi) = f \cup_\psi (g \cup_\psi)$$

$F^+(U)$ param by $X(\Delta), z$ w) $\bar{z} = \frac{dx}{ds}$

$$\mathcal{X}_{\text{cm}} = \{ \text{Hopf alg gen by } \frac{dx}{ds} \} \rtimes U(\mathfrak{g})$$

$$\mathfrak{g} = \text{gen by } (0, \frac{\partial}{\partial z}), \frac{d}{ds}$$

$$\bar{z} = (\frac{dx}{ds}, \frac{dz}{ds})$$

left coaction, $Y \rightarrow Y \otimes 1, X \rightarrow X \otimes 1 + \delta_{1,0}$

right action $[X, \delta_n] = \delta_{n+1}, [Y, \delta_n] = n \delta_n$

$$\mathcal{X} = \mathbb{Q}[\delta_n] \hookrightarrow \mathcal{X} \text{ Connes-Kreimer 99}$$

We want $HH^1(\mathcal{X}, \mathcal{X})$, not the smaller space!

consider $\frac{d}{ds} \bar{x} = \bar{f}(x) \rightarrow \frac{\partial \bar{x}^i}{\partial s} = \bar{f}^i$

$$\varphi^i \mathcal{X} \rightarrow C_c^\infty(M)$$

$$t \rightarrow \prod_{v \in V(E)} \frac{\partial^{\text{left}(v)}}{\prod \partial x_{i_v}} f^{\text{parent}(v)} \text{ if } v = \text{root, } \Rightarrow i_{\text{parent}(v)} = j$$

e.g. φ^i : $\rightarrow \frac{\partial f^i}{\partial x^j \partial x^k} f^j f^k$

New grafting operator

$$\varphi^i(u) \frac{\partial}{\partial x} \varphi^j(t) = \varphi^i(N_u(t)) = \sum_{v \in V(t)} \text{hang u off vert v of t}$$

Eg: $t = \mathcal{R}$ $u = \mathcal{L}$ $N_u(t) = 2 \mathcal{R} + \mathcal{L}$

- 1) Pre-Lie.
- 2) $N_u(t) \rightarrow f \frac{\partial}{\partial x} \varphi^i(t) = \frac{\partial}{\partial s} \varphi^i(t)$ Nat growth.
- 3) New Hopf alg $\mathbb{Q}[\varphi^i(t) | t \in \mathcal{X}_0] \rtimes U(\mathfrak{g}) =: \bar{\mathcal{X}}_{\text{cm}}$

$$\frac{\partial}{\partial s}(x, z) = (f^i, f^j)$$

$$g = (0, \frac{\partial}{\partial z}) \quad X_t = \varphi^i(t) \frac{\partial}{\partial x} + \varphi^j(t) \frac{\partial}{\partial z}$$

Q: Calculate $HC^1(\bar{\mathcal{X}}_{\text{cm}})$ via action on A .
What does this say about $HH^1(\bar{\mathcal{X}}, \bar{\mathcal{X}})$

Other interesting structures?