Combinatorial DSE.

Def Hopf algebra of Rooted trees

$\mathcal{H} = \mathbb{Q}[$ rooted tree, oriented $\downarrow]$

Prop: prod: disjoint union

admiss cut: collection of edges, $c \in E(T)$ s.t. any path contains at most 1

$\Delta_c(T) = \sum P(c) \otimes R(c)$

- Full cut
- Empty cut

$R(c) =$, $P(c) = \odot$

$$\Delta T = \sum_{\text{admiss cuts}} \Delta_c(T)$$

$B_+ : \mathcal{H} \to \mathcal{H}$ Linear operator.
forest $\to$ tree

Eg of Combinatorial DSE:

① $X = 1 + \alpha B_+(X^2) =$

$1 + \alpha \bullet + 2\alpha^2 + \alpha^3(2 + \wedge) + \alpha^4(2 + + 4)$

Partial Green's function

$\Gamma = 1 + \alpha + 2\alpha^2 + \alpha^3(+) + \text{etc}$

In number theory:

$X = 1 + \alpha B_+(X) = 1 + \alpha \bullet + \alpha^2 + \alpha^3 + \cdots$

$$\longrightarrow \int_0^z \frac{dt_n}{t_n} \cdots \int_0^{t_3} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{1-t_1}$$

$Li_n(z)$

$\Rightarrow X = \sum \alpha^n Li_n(z)$

More examples, see Manin

Eg computation Theory.

Note: $\Delta B_+(t_1 \ldots t_k) = \sum_{c \text{ admiss}} P(c) \otimes B_+(R(c)) + B_+(t_1 \ldots t_k) \otimes 1$

doesn't take full total cut

Connection to non-commutative geometry

Hochschild cohomology

A = associative alg

$T(A) = \bigoplus_{n \geq 1} A^{\otimes n}$

Hochschild homology: $\partial : A^{\otimes n} \to A^{\otimes n-1}$

$[a_1 \otimes a_2 \ldots a_n] \to \sum (-1)^i a_1 \otimes \ldots a_i a_{i+1} \ldots a_n$

de Rham cohom in non-commutative setting

Hochschild cohom:

$\Delta^{(i)} : A^n \to A^{n+1}$

$a_1 \otimes \ldots a_n \to a_1 \otimes \ldots \otimes \Delta a_i \otimes \ldots a_n$

$C^{\bullet}(B, A) = (\mathrm{Hom}_{\mathrm{lin}}(B, A^{\otimes \bullet}), b)$

$T \in C^n$

$bT = (1 \otimes T)\Delta + \sum (-1)^i \Delta^{i}(T) + T \otimes 1$

Thm $B_+ \in HH^1(\mathcal{H}, \mathcal{H})$ (Kreimer, Bergbauer)

Thm (Hochschild–Kostant–Rosenberg)

$HH^1(\mathcal{H}, \mathcal{H})$ = vector space generated by prim graphs

Def Prim graph: $\Delta t = 1 \otimes t + t \otimes 1$

Eg $\bullet$, $3(\cdot\, | - \,) - \ldots$

etc.

However, we know only 1 element!

Q? What others exist?
What is the cohomology?

Where has this been studied?

Connes–Moscovici (98)

M = 1-dim locally param mfld

$\psi : U \to U$, $s \in \mathbb{R}$, M

$A = C_c^\infty(F^+(M)) \rtimes \mathrm{Diff}^+(M)$

$f U_\psi$, $U \subset M$ defines supp f

$(f U_\psi)(g U_\varphi) = f(\psi_* g) U_{\varphi \circ \psi}$

$F^+(U)$ param by $x(s), z$ w/ $\dot{z} = \frac{dx}{ds}$

$\mathcal{H}_{CM} = \{\text{Hopf alg gen by } \frac{d^n x}{ds^n}\} \bowtie \mathcal{U}(g)$

g = gen by $(0, \frac{\partial}{\partial z})$, $\frac{d}{ds}$

Y, X

$\delta = \left(\frac{dx^n}{ds^n}, \frac{d^2 z}{d s^2}\right)$

left coaction, $Y \to Y \otimes 1$, $X \to X \otimes 1 + \delta_1 \otimes Y$

right action $[X, \delta_n] = \delta_{n+1}$, $[Y, \delta_n] = n\delta_n$

$\mathcal{H} = \mathbb{Q}[\delta_n] \longleftrightarrow \mathcal{H}$ Connes–Kreimer 99

We want $HH^1(\mathcal{H}, \mathcal{H})$, not the smaller space!

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consider $\frac{d}{ds}\vec{x} = \vec{f}(x) \to \frac{\partial x^i}{\partial s} = f^i$

$\varphi^i : \mathcal{H} \to C_c^\infty(M)$

$t \longrightarrow \prod_{v \in V(t)} \frac{\partial^{f(v)}}{\prod_{v' \text{ daughter}} \partial x_{i_{v'}}} f^{i_{\text{parent}(v)}}$

if v = root, $\Rightarrow i_{\text{parent}(v)} = j$

e.g: φ^i: f^i_{jk}, f^j, f^k, f^l $\longrightarrow \frac{\partial f^i}{\partial x^j \partial x^k} f^j \frac{\partial f^k}{\partial x_l} f^l$

New grafting operator

$\varphi^i(u) \frac{\partial}{\partial x^j} \varphi^j(t) = \varphi^i(N_u(t)) = \sum_{v \in V(t)}$ hang u off vert v of t

Eg $t = $, $u = $, $N_u(t) = 2 + $

① Pre-Lie

② $N_\bullet(t) \to f^j \frac{\partial}{\partial x^j} \varphi^i(t) = \frac{\partial}{\partial s} \varphi^i(t)$ Nat growth

③ New Hopf alg

$\mathbb{Q}[\varphi^i(t) | t \in \mathcal{H}_c] \bowtie \mathcal{U}(g) =: \overline{\mathcal{H}_{CM}}$

$\frac{\partial}{\partial s}(x, z) = (f^i, f^i)$

$g = (0, \frac{\partial}{\partial z})$ $X_t = \varphi^i(t) \frac{\partial}{\partial x} + \varphi^i(t) \frac{\partial}{\partial z}$

Q: Calculate $HC^{\bullet}(\overline{\mathcal{H}}_{CM})$ via action on A.

What does this say about $HH^1(\overline{\mathcal{H}}, \overline{\mathcal{H}})$

Other interesting structures?